Notes on random fields

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Setting: Definitions and notations

(*i*) Let $\Omega := \mathbb{R}^{\mathbb{Z}^d}$ be the *configuration space* and let $\Omega_\Lambda := \mathbb{R}^\Lambda$ be the *finite volume configuration space* for a finite subset Λ of $\mathbb{Z}^d$ which we denote by $\Lambda \Subset \mathbb{Z}^d$. We will endow them with the product σ field $\mathcal{F}$ and $\mathcal{F}_\Lambda$ respectively. The elements of Ω and Ω_Λ are called *configurations* or *field*.

(*ii*) A *random field* on $\mathbb{Z}^d$ or Λ is a probability measure on Ω or Ω_Λ respectively and we write $\mu \in \mathcal{M}_1(\Omega)$ or $\mu \in \mathcal{M}_1(\Omega_\Lambda)$. Sometimes we will not distinguish between the measure μ and a field ϕ drawn according to μ.

(*iii*) Further the *finite volume Hamiltonian* or *free energy* of a configuration $\phi \in \Omega$ as

$$H_{\Lambda,m}(\phi) := \frac{1}{4d}\sum_{x\sim y}\left(\phi_x - \phi_y\right)^2 + \frac{m^2}{2}\sum_{x\in\Lambda}\phi_x^2 =: \frac{1}{4d}(\nabla\phi,\nabla\phi)_\Lambda + \frac{m^2}{2}(\phi,\phi)_\Lambda$$

where the first sum is taken over all neighboring sides x and y (denoted by $x \sim y$) such that x or y is contained in Λ. The quantity $m \geq 0$ is called the *mass* of the system and in the case $m = 0$ the system is called *massless* and *massive* for $m > 0$.

(*iv*) The *finite volume Gibbs measure* with *boundary value* $\eta \in \Omega$ is defined as the probability measure on Ω such that

$$\gamma^\eta_{\Lambda,m}(A) \propto \int\limits_{\mathbb{R}^\Lambda} e^{-H_{\Lambda,m}(\phi)}\chi_A(\phi_\Lambda\eta_{\Lambda^c})\prod_{x\in\Lambda}\mathrm{d}\phi_x$$

where $\phi_\Lambda\eta_{\Lambda^c}$ denotes the field which is obtained by changing the values of ϕ outside of Λ to the ones of η. Obviously the Gibbs measure $\gamma^\eta_{\Lambda,m}$ is supported by the set of configurations which agree with η outside of Λ, which explains the terminology of boundary conditions.

(*v*) We say a random field $\mu \in \mathcal{M}_1(\Omega)$ is an *infinite volume Gibbs measure* if it satisfies the consistency condition

$$\mu(A \mid \mathcal{F}_{\Lambda^c})(\eta) = \gamma^\eta_{\Lambda,m}(A) \quad \text{for all } \Lambda \Subset \mathbb{Z}^d.$$

A field ϕ can be seen as a random surface over $\mathbb{Z}^d$. The Hamiltonian of a configuration should be thought of as a measurement of internal energy of a configuration ϕ and the first term of the Hamiltonian penalises high differences between neighboring points, the second one penalises high absolut values of the field. It is intuitive that those two phenomena are energetically unfarourable in the sense that a surface tends to avoid being stretched and also tends to stay rather close to the ground(?) if it is made out of massive material. The finite volume Gibbs measure implements the physical principle that high energy states are unlikely to occur because it assigns a low probability to the configurations with a high free energy. The consistency condition of a infinite volume Gibbs measure should be thought of as that if we are given that the field coincides outside of Λ with η then we recover the finite volume Gibbs measure.

In the following we will investigate the existence of infinite volume Gibbs measures which are closely related to the reccurence and transienity of certain random walks. Further we will give a probabilistic solution of the discrete Dirichlet boundary problem using this same random walk.

Thermodynamic limit

We will investigate the limiting behaviour of Gaussian measures on finite volume subspaces $\Omega_\Lambda \subset \Omega$, where each finite volume configuration is just extended onto $\mathbb{Z}^d$ by setting it zero outside of Λ. In an analogue manner we interpret a random field on Λ as a random field on $\mathbb{Z}^d$. Before we can state the theorem we shall note that we call a measure *Gaussian*, if all his finite dimensional marginals are Gaussian. Further we say it has mean $a \in \Omega$ and covariance $\Sigma \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ if for every finite subset $\Lambda \Subset \mathbb{Z}^d$ the finite dimensional projection onto Λ has mean a_Λ and covariance Σ_Λ where they denote the restriction of the vector and matrix respectively.

1 Theorem (Thermodynamic limit)**.** *Let B_n be the n-ball in $\mathbb{Z}^d$ and let $\mu_n \in \mathcal{M}_1(\Omega_{B_n})$ be Gaussian measures with mean a_{B_n} and covariance structure Σ_{B_n}. Assume further*

$$a_{B_n} \to a \quad \text{and } \Sigma_{B_n} \to \Sigma$$

entrywise. Then the following two statements hold.

(*i*) *There is a unique Gaussian field μ on $\mathbb{Z}^d$ with mean a and covariance Σ, i.e. the finite dimensional projections μ_Λ onto $\Lambda \Subset \mathbb{Z}^d$ are Gaussian with mean a_Λ and covariance Σ_Λ.*

(*ii*) *The finite dimensional projections of μ_n converge weakly towards the one of μ, i.e.*

$$(\mu_n)_\Lambda \to \mu_\Lambda \quad \text{weakly for all } \Lambda \Subset \mathbb{Z}^d.$$

Proof. The first claim follows just from Kolmogorovs extension theorem and the apparent fact that the declaration of the finite dimensional marginals is consistent. The second statement is due to the convergence of the mean and covariance matrix and can be checked again by taking the Fourier transforms of the measures. □

In the following we will show that the finite volume Gibbs measures $\gamma^\eta_{\Lambda,m}$ are Gaussian with mean u_Λ and covariance G_Λ where u is the so called *m-harmonic extension* of η, i.e. it solves the *massive Dirichlet problem*

$$\left(-\frac{1}{2d}\Delta + m^2\right) u = 0 \quad \text{in } \Lambda \quad \text{and } u = \eta \quad \text{in } \Lambda^c$$

and G_Λ is the *Greens function* of a suitable random walk. Note that the *discrete Laplace operator* is given by

$$(\Delta u)_x \coloneqq \sum_{y\sim x} \big(u_y - u_x\big).$$

The study of those Greens functions and their convergence for $\Lambda \to \mathbb{Z}^d$ can then be used to either construct infinite volume Gibbs measures (if the Greens function converges) or show the non existence of infinite volume Gibbs measures (if it blows up).

The massless case

We defined the finite volume Gibbs measure over its Lebesgue density and in order to show that it is Gaussian we need to rewrite that density as a normal density, which we will achieve by convert the Hamiltonian as a quadratic function in ϕ. Precisely we note that we obtain for a harmonic extension u of η (the existence of such a u will be shown later)

$$\frac{1}{2d}(\nabla\phi,\nabla\phi)_\Lambda = (\phi_\Lambda - u_\Lambda)\cdot\left(-\frac{1}{2d}\Delta_\Lambda\right)(\phi_\Lambda - u_\Lambda) + B(\eta)$$

via partial integration, where the boundary terms only depend on the boundary terms of u and of ϕ which are $\gamma^\eta_{\Lambda,0}$ almost surely η. The *discrete Laplace matrix* Δ_Λ is the restriction of the discrete Laplace matrix on $\mathbb{Z}^d$, which is given by

$$\Delta(x,y) \coloneqq \begin{cases} -2d & \text{if } x = y \\ 1 & \text{if } x\sim y \\ 0 & \text{otherwise} \end{cases}.$$

Note that we do not have to care about the boundary term $B(\eta)$ since it will cancel out in the normalisation of the measure. Assume for one second that the matrix $-\frac{1}{2d}\Delta_\Lambda$ has in inverse G_Λ, then we would know that the Gibbs measure is Gaussian with mean u_Λ and covariance G_Λ. To convince ourselves that the inverse actually exists, we note $-\frac{1}{2d}\Delta_\Lambda = I - P_\Lambda$ where P is the transition matrix of a *simple symmetric random walk* (SSRW) on $\mathbb{Z}^d$, i.e.

$$P(x,y) \coloneqq \begin{cases} \frac{1}{2d} & \text{if } x\sim y \\ 0 & \text{otherwise} \end{cases}.$$

Further we have $\|P_\Lambda\| < 1$ and thus the inverse, called the Greens function, G_Λ exists and is given by the Neumann series

$$G_\Lambda = \sum_{n\geq 0} P^n_\Lambda.$$

Evaluating this at (x,y) this yields

$$G(x,y) = \sum_{n\geq 0} \mathbb{P}_x(X_n, \tau_{\Lambda^c} < n) = \mathbb{E}_x\Big[\#\left\{n\geq 0 \mid X_n = y, \tau_{\Lambda^c} < n\right\}\Big]$$

where (X_n) is a SSRW and τ_{Λ^c} is the exit time from Λ. This means that the value of the Greens function $G(x, y)$ is the expected number of visits of a SSRW in y if it starts in x and is killed when it hits the boundary of Λ.

So far we have assumed the existence of a harmonic extension u of η onto Λ and postponed the prove. Interestingly u can also be constructed through the SSRW by setting

$$u_x := \mathbb{E}_x \left[\eta_{X_{\tau_{\Lambda^c}}} \right].$$

Obviously u is an extension of η and further it is harmonic in Λ by

$$u_x = E_x \left[\mathbb{E}_x \left[\eta_{X_{\tau_{\Lambda^c}}} \middle| X_1 \right] \right] = \sum_{y \sim x} \frac{1}{2d} \mathbb{E}_y \left[\eta_{X_{\tau_{\Lambda^c}}} \right] = \sum_{y \sim x} \frac{1}{2d} u_y,$$

where we used the Markov property.

Since the procedure will be analogue in the massive case, we shall shortly recall the main steps of our argument:

(i) Rewrite the Hamiltonian as a quadratic form – modulo the existence of a harmonic extension.

(ii) Invert the matrix to conclude that the Gibbs measure is Gaussian, this is done by the so called *random walk representation*.

(iii) Solve the Dirichlet problem with boundary values η – again through the random walk represenation. This shows that step one was possible.

The theory of random walks tells us now

$$G_{B_n}(0,0) \approx \begin{cases} n & \text{for } d = 1 \\ \log(n) & \text{for } d = 2 \end{cases}$$

and further it converges for $d \geq 3$ since the SSRW is transient in that case. Those results can be used to show the non existence of a infinite volume Gibbs massless measure in dimension $d = 1, 2$ and the existence of infinitely man Gaussian Gibbs measures in dimension $d \geq 3$. Further note that the behaviour of the Greens function in one dimension should come as no big suprise, since Donskers invariance principle tells us, that a (pinned down) random walk has to be scaled by $\frac{1}{\sqrt{n}}$ to obtain a non trivial but well defined limiting object.

The massive case

As already mentioned the approach is completely analogue to the massless case with only some minor adjustments. More precisely for a m-harmonic extension u of η we get

$$2 \cdot H_{\Lambda,m}(\phi) = (\phi_\Lambda - u_\Lambda) \cdot \left(-\frac{1}{2d} \Delta_\Lambda + m^2 \right) (\phi_\Lambda - u_\Lambda) + B(\eta).$$

Now we have

$$-\frac{1}{2d} \Delta_\Lambda + m^2 = (1 + m^2) I - P_\Lambda = (1 + m^2) \big(I - \hat{P}_\Lambda \big)$$

where $\hat{P} = (1+m^2)^{-1}P$ is the transition matrix of a SSRW that gets killed randomly with probability $\frac{m^2}{1+m^2}$. Thus the covariance is now given by

$$G(x,y) = \frac{1}{1+m^2}\sum_{n\geq 0}\mathbb{P}_x(Y_n, \tau_{\Lambda^c} < n) = \frac{1}{1+m^2}\mathbb{E}_x\Big[\#\left\{n \geq 0 \mid Y_n = y, \tau_{\Lambda^c}\right\}\Big]$$

where (Y_n) is a SSRW with random killing. Further it can easily be checked that the m-harmonic extension of η is given by

$$u_x := \mathbb{E}_x\left[\eta_{Y_{\tau_{\Lambda^c}}}\right].$$

Since the SSRW with a positive killing probability is transient in every dimension, the Greens function converges always and this can be used to construct a Gaussian field on the whole lattice $\mathbb{Z}^d$ which is a weak limit of Gibbs measures and it can be shown that it is Gibbsian again. This method is again completely analogue to the massless case (where it is only applicable if $d \geq 3$). Further the random walk representation can be used to show a CLT like theorem.

2 Theorem (CLT for lattice structure)**.** *Let μ be a centered and massive infinite volume Gibbs measure. Further let $\Lambda_N := (-N,N]^d$ be the N-box and let $\phi \sim \mu$. Then one has*

$$\frac{1}{(2N)^{d/2}}\cdot\sum_{x\in\Lambda_N}\phi_x \to \mathcal{N}(0,m^{-2}).$$

Proof. Obviously the random variables on the left hand side are all centered Gaussians and therefore we only have to check the convergence of the variance. Note that it is enough for that to show

$$\sum_{x\in\mathbb{Z}^d}\mathbb{E}_\mu[\phi_0\phi_x] = m^{-2}.$$

However this holds since the left side equals

$$\sum_{x\in\mathbb{Z}^d}G(0,x) = \sum_{x\in\mathbb{Z}^d}\mathbb{E}_0\Big[\#\left\{n\geq 0 \mid X_n = y, \tau_\star < n\right\}\Big] = \mathbb{E}_0[\tau_\star] = m^{-2},$$

where $\tau_\star$ is the time the random walk gets killed. □

The above CLT like result is actually not really surprising in the light that one can show that the correlation between the ϕ_x and ϕ_y (which is just $G(x,y)$) decays exponentially with $\|x-y\|\to\infty$ and thus this is only a CLT for weakly correlated random variables.