Weak compactness of probability measures

Johannes Müller

February 7, 2019

Abstract

We will investigate fundamental properties of finitely additive signed measures, like the Hahn-Jordan decomposition and the integration with respect to them. This notion of integral leads to a canonical representation of measures as a dual space, namely the dual of all bounded measurable functions. This approach has the advantage that it immediately gives the Banach space property of the signed measures with respect to the total variation. Further we will prove that the countably additive measures correspond exactly to the dual elements for which the desirable monotone and dominated convergence theorem hold. This characterisation naturally leads to the characterisation of dual spaces of other Banach spaces of bounded functions like spaces of continuous functions.

Let X be an arbitrary set and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. Further let $L_b^0(\mathcal{A})$ be the space of real measurable and bounded functions on X which is a Banach space with respect to the uniform norm $\|\cdot\|_\infty$. The completeness of $L_b^0(\mathcal{A})$ follows from the fact that the measurable bounded functions are closed in the larger Banach space of bounded functions as they are even closed under pointwise convergence, or more precisely the product topology. We will see that the finitely additive measures on $\mathcal{A}$ are just the dual space of $L_b^0(\mathcal{A})$, but before we can prove this we need a few basic properties of signed measures.

1 Definition (Signed measure). Let $(X, \mathcal{A})$ be a measurable space and let $\mu\colon \mathcal{A} \to \mathbb{R}$. Then we say μ is

(i) *finitely additive* if we have
$$\mu\Big(\bigcup_{k=1}^N A_k\Big) = \sum_{k=1}^N \mu(A_k)$$
for all finite collection of measurable disjoint sets $A_k \in \mathcal{A}$,

(ii) *countably additive* if the above holds for countable collections of measurable disjoint sets,

(iii) of *bounded variation* or shortly *bounded* if we have
$$\|\mu\|_{BV} := \sup_{\mathcal{E}} \sum_{E\in\mathcal{E}} |\mu(E)| < \infty$$
where the supremum is taken over all finite families of disjoint measurable sets,

(iv) *positive* if $\mu(\mathcal{A}) \subseteq \mathbb{R}_+$.

Further we denote the space of bounded and finitely additive measures by $ba(\mathcal{A})$ and the space of bounded and countably additive measures by $ca(\mathcal{A})$. The quantity $\|\mu\|_{BV}$ is called the norm of *total variation* of μ and Theorem 6 shows that it is indeed a norm and that both $ba(\mathcal{A})$ and $ca(\mathcal{A})$ are complete wrt to it.

2 Theorem (Hahn-Jordan decomposition). *For every signed measure $\mu \in ba(\mathcal{A})$ there are two positive measures $\mu_+, \mu_- \in ba(\mathcal{A})$ such that $\mu = \mu_+ - \mu_-$. If μ is countably additive then μ_+ and μ_- can be chosen to be countably additive as well.*

Proof. For $A \in \mathcal{A}$ set
$$\mu_+(A) := \sup\left\{\mu(B) \mid B \in \mathcal{A}, B \subseteq A\right\}.$$
It is clear that μ_+ is positive as we have $\mu_+(A) \geq \mu(\varnothing) = 0$. To see that μ_+ is finitely additive let A and B be disjoint measurable sets. Then we have

$$\begin{aligned}
\mu_+(A \cup B) &= \sup\left\{\mu(C) \mid C \in \mathcal{A}, C \subseteq A \cup B\right\} \\
&= \sup\left\{\mu(C \cap A) + \mu(C \cap B) \mid C \in \mathcal{A}, C \subseteq A \cup B\right\} \\
&= \sup\left\{\mu(C) + \mu(D) \mid C, D \in \mathcal{A}, C \subseteq A, D \subseteq B\right\} \\
&= \sup\left\{\mu(C) \mid C \in \mathcal{A}, C \subseteq A\right\} + \sup\left\{\mu(D) \mid D \in \mathcal{A}, D \subseteq B\right\} \\
&= \mu_+(A) + \mu_+(B).
\end{aligned}$$

If we choose $\mu_- := \mu_+ - \mu$ we get the desired decomposition. Note that μ_- is positive since we have $\mu \leq \mu_+$ via definition.

Let now μ be countably additive then it suffices to show that μ_+ is countably additive so let $(A_n) \subseteq \mathcal{A}$ be a sequence of disjoint sets. Choose now N so large that

$$\sum_{i=N+1}^{\infty} \mu_+(A_i) < \varepsilon.$$

To see that such N exists, take sequences $(B_n^i)_n \subseteq \mathcal{A}$ such that $B_n^i \subseteq A_i$ and $0 \leq \mu(B_n^i) \nearrow \mu_+(A_i)$. We get now

$$\|\mu\|_{BV} \geq \sum_{i=1}^{\infty} \mu(B_n^i) \nearrow \sum_{i=1}^{\infty} \mu_+(A_i) \quad \text{for } n \to \infty$$

by monotone convergence. Let $\varepsilon > 0$ and choose $B \subseteq A := \bigcup_{i=1}^{\infty} A_i$ such that

$$\mu(B) \geq \mu_+(A) - \varepsilon.$$

With $A^N := \bigcup_{i=1}^{N} A_i$ we get

$$\mu_+(A^N) \geq \mu(B \cap A^N) = \mu(B) - \mu\big(B \cap (A \setminus A^N)\big) \geq \mu_+(A) - 2\varepsilon$$

since we can estimate

$$\mu\big(B \cap (A \setminus A^N)\big) = \sum_{i=N+1}^{\infty} \mu(B \cap A_i) \leq \sum_{i=N+1}^{\infty} \mu_+(A_i) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we get

$$\sum_{i=1}^{N} \mu_+(A_i) = \mu_+(A^N) \to \mu_+(A) \quad \text{for } N \to \infty.$$

□

Integration with respect to a signed measure

The integral of a *simple function* with respect to a finitely additive measure is defined in the familiar way

$$\int \sum_{k=1}^{N} \alpha_i \chi_{A_i} \, \mathrm{d}\mu := \sum_{k=1}^{N} \alpha_i \mu(A_i).$$

It is easily seen that this definition does not depend on the representation as a finite sum. Note that the space of simple functions

$$\mathcal{E} = \left\{ \sum_{k=1}^{N} \alpha_i \chi_{A_i} \,\middle|\, \alpha_i \in \mathbb{R}, A_i \in \mathcal{A}, N \in \mathbb{N} \right\}$$

is a dense subspace of $L_b^0(\mathcal{A})$ with respect to the uniform norm. Given $f \in L_b^0(\mathcal{A})$ an approximating sequence would be given by

$$f_n = \sum_{k=-2^n}^{2^n+1} k \,\|f\|\, 2^{-n} \chi_{A_k} \quad \text{with } A_k = f^{-1}\big([k \,\|f\|\, 2^{-n}, (k+1)\,\|f\|\, 2^{-n})\big).$$

If we write $f = \sum \alpha_i \chi_{A_i} \in \mathcal{E}$ such that the sets A_i are pairwise disjoint we get

$$\left| \int f \,\mathrm{d}\mu \right| \leq \sum_{k=1}^{N} |\alpha_i| \cdot |\mu(A_i)| \leq \max_i |\alpha_i| \cdot \sum_{k=1}^{N} |\mu(A_i)| \leq \|f\|_\infty \cdot \|\mu\|_{BV} \,. \tag{1}$$

Therefore $\int \mathrm{d}\mu \colon \mathcal{E} \to \mathbb{R}$ is a bounded linear operator and can uniquely be extended to a bounded linear operator on the whole space $L_b^0(\mathcal{A})$ which we will denote again by $\int \mathrm{d}\mu$ in abusive notation. Further the inequality (1) carries over to all functions $f \in L_b^0(\mathcal{A})$, so we have proven the following result.

3 Theorem (Integral). *For every signed measure $\mu \in ba(\mathcal{A})$ there is a unique linear and continuous mapping*

$$\int \mathrm{d}\mu \colon L_b^0(\mathcal{A}) \to R$$

called the integral wrt to μ *such that we have*

$$\int \sum_{k=1}^{N} \alpha_i \chi_{A_i} \mathrm{d}\mu := \sum_{k=1}^{N} \alpha_i \mu(A_i)$$

for all simple functions. Further the estimate (1) *holds for all $f \in L_b^0(\mathcal{A})$.*

4 Remark. From the definition of the integral we immediately get

$$\int \mathrm{d}\mu = \int \mathrm{d}\mu_+ - \int \mathrm{d}\mu_-$$

for simple functions and this decomposition carries over to the general integral. Considering this identity, we see immediately that all the nice properties of the Lebesgue integral like the monotone and dominated convergence theorem carry over to the integral with respect to a signed countably additive measure.

5 Remark. The class of integrands is fairly small and it is well known from measure and integration theory that for countably additive (signed) measures the notion of integral can be extended to a much wider class of functions. However one needs the monotone convergence theorem (which only holds for countably additive measures) to show that this more general Lebesgue integral is linear.

Measures as dual spaces

Now we have set up all the preliminaries that we need to perceive $ba(\mathcal{A})$ as a dual space.

6 Theorem (Measures as a dual space). *With the above notations the linear mapping*

$$I : ba(\mathcal{A}) \to L_b^0(\mathcal{A})^*, \quad \mu \mapsto I\mu := \left(f \mapsto \int f \,\mathrm{d}\mu \right)$$

is an algebraic isomorphism and we have $\|I\mu\|_{L_b^0(\mathcal{A})^*} = \|\mu\|_{BV}$. *In particular* $\|\cdot\|_{BV}$ *is a norm and* $ba(\mathcal{A})$ *is complete with respect to it. Further the inverse* I^{-1} *is given by*

$$F \mapsto \Big(A \mapsto F(\chi_A)\Big).$$

For a measure $\mu \in ba(\mathcal{A})$ *the following three statements are equivalent:*

(i) μ *is countably additive.*

(ii) $I\mu$ *fulfills the monotone convergence theorem.*

(iii) $I\mu$ *fulfills the dominated convergence theorem.*

Proof. The previous theorem directly implies that I is a well defined linear contraction, i.e. $\|I\mu\| \leq \|\mu\|$. To see that the mapping is an isometry we take a finite collection $\mathcal{E}$ of disjoint measurable sets. Then we get

$$\|I\mu\|_{L_b^0(\mathcal{A})^*} \geq \int \sum_{E\in\mathcal{E}} \operatorname{sign}(\mu(E))\chi_E \mathrm{d}\mu = \sum_{E\in\mathcal{E}} |\mu(E)|\,.$$

By taking the supremum over all those collections $\mathcal{E}$ we get $\|I\mu\| = \|\mu\|$. Further the bijectivity follows from the fact $I \circ I^{-1} = \mathrm{id}_{L_b^0(\mathcal{A})^*}, I^{-1} \circ I = \mathrm{id}_{ba(\mathcal{A})}$.

Let now μ be countably additive, then we know, that (*ii*) and (*iii*) hold, so we only have to show that (*ii*) and (*iii*) both imply (*i*). If we assume that the monotone convergence theorem holds for $I\mu$, we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = I\mu\left(\sum_{n\in\mathbb{N}} \chi_{A_n}\right) = \sum_{n\in\mathbb{N}} I\mu(\chi_{A_n}) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for disjoint $(A_n) \subseteq \mathcal{A}$. Thus μ is countably additive. The implication (*iii*)$\Rightarrow$(*i*) follows analogue. □

7 Remark. From now on we will identify μ and $I\mu$ with each other and therefore we write $\mu(f)$ for the integral of f wrt to μ whenever this makes sense.

8 Corollary. *The space of countably additive signed measures is a Banach space with respect to the norm of total variation.*

Proof. Since $ba(\mathcal{A})$ is complete, we only have to prove that $ca(\mathcal{A})$ is closed. For this let $(\mu_n) \subseteq ca(\mathcal{A})$ be a sequence with $\mu_n \to \mu$. To see that μ is again countably additive we only have to show that the dominated convergence theorem holds for μ. So let $(f_n) \subseteq L_b^0(\mathcal{A})$ be a sequence of functions such that $f_n \to f$ pointwise and $\sup_n |f_n| \leq K < \infty$. The computation

$$\begin{aligned}\lim_{n\to\infty} |\mu(f_n) - \mu(f)| &\leq \liminf_{n\to\infty} |\mu(f_n) - \mu_N(f_n)| + |\mu_N(f_n) - \mu_N(f)| \\ &\quad + |\mu_N(f) - \mu(f)| \\ &\leq 2K \cdot \|\mu - \mu_N\|_{BV} \to 0 \quad \text{for } N \to \infty\end{aligned}$$

completes the proof. □

The interpretation of measures as the dual space of bounded measurable functions can be used to show that the dual space of $L^\infty(\mu)$ coincides with the absolutely continuous measures wrt μ. In the case of signed measures a measurable set $A \in \mathcal{A}$ is called a μ *Null set* if $\mu_+(A) + \mu_-(A) = 0$. A measure $\nu \in ba(\mathcal{A})$ is said to be *absolutely continuous* wrt μ if every μ Null set is also a ν Null set. Further we write $L^\infty(\mu)$ for the space of all measurable function that are bounded outside of a μ Null set which is a Banach space wrt to the usual $L^\infty(\mu)$ norm.

9 Theorem (Dual of $L^\infty(\mu)$). *Let* $\mu \in ba(\mathcal{A})$ *and let* $ba(\mu) \subseteq ba(\mathcal{A})$ *be the subspace of measure that are absolutely continuous wrt* μ. *Then we have*

$$ba(\mu) \cong L^\infty(\mu)^*$$

isometrically in the canonical way.

Proof. Let $[f]_\mu$ denote the μ ,a.e. equivalence' class of $f \in L_b^0(\mathcal{A})$, then

$$\iota : L_b^0(\mathcal{A}) \to L^\infty(\mu), \quad f \mapsto [f]_\mu$$

is surjective and further we have

$$\left\| [f]_\mu \right\|_{L^\infty(\mathcal{A})} = \inf_{g \in [f]_\mu} \|g\|_\infty . \tag{2}$$

Therefore the dual operator

$$\iota^* \colon L^\infty(\mu)^* \to L_b^0(\mathcal{A})^* \cong ba(\mathcal{A})$$

is injective and because of (2) it is isometric since it implies $\iota(B_1) = B_1$ where B_1 denotes the unit ball in the respective spaces. Further the closed range theorem implies

$$\operatorname{ran}(\iota^*) = \ker(\iota)^\perp = \left\{ F \in L_b^0(\mathcal{A})^* \mid F(f) = 0 \text{ for all } f = 0\ \mu \text{ a.e.} \right\} \cong ba(\mu).$$

□

10 Characterisation of other dual spaces. A similar approach can be used to characterise the dual spaces of other normed spaces of bounded measurable functions, for example the dual of all bounded continuous functions $\mathcal{C}_b(X)$ if we are dealing with a topological space X. If we denote the Borel algebra by $\mathcal{B}$ we get

$$\iota \colon \mathcal{C}_b(X) \hookrightarrow L_b^0(\mathcal{B})$$

isometrically and therefore

$$\iota^* \colon ba(\mathcal{B}) \cong L_b^0(\mathcal{B})^* \twoheadrightarrow \mathcal{C}_b(X)^*, \quad \mu \mapsto \big(f \mapsto \mu(f)\big)$$

is surjective, but not necessarily isometric and not injective if $\mathcal{C}_b(X) \subsetneq L_b^0(\mathcal{B})$. Of course one could argue that we then have $\mathcal{C}_b(X)^* \cong ba(\mathcal{B})/\ker(\iota^*)$ but that characterisation is very implicit. However one can characterise that quotient space is some cases – usually under certain regularity condition on the space – as a subspace of $ba(\mathcal{A})$. For detail on this we refer to the chapter about positive functionals in [Rudin, 2006].

Conclusion

We have seen that the Hahn-Jordan decomposition also holds for only finitely additive signed measures. To the authors knowledge there is no reference so far for this in the literature. Further we developed a duality between a function space and the Banach space of signed measures and proved that countable additivity is equivalent to the dominated or monotone convergence theorem. This duality provides an extremely short and self contained introduction to signed measures that avoids the mostly tedious calculation that are usually involved in the proof of the norm property of the total variation and the completeness of the space of signed measures.

References

[Billingsley, 2013] Billingsley, P. (2013). *Convergence of probability measures*. John Wiley & Sons.

[Klenke, 2013] Klenke, A. (2013). *Probability theory: a comprehensive course*. Springer Science & Business Media.

[Müller, 2018] Müller, J. (2018). Functional analytic crashcourse in topology. `https://github.com/muellerjohannes/functional-analytic-crash-course`.

[Rudin, 1991] Rudin, W. (1991). Functional analysis. 1991. *Internat. Ser. Pure Appl. Math.*

[Rudin, 2006] Rudin, W. (2006). *Real and complex analysis*. Tata McGraw-Hill Education.