ODEs in Frechét spaces

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Abstract

The famous Picard-Lindelöf theorem yields local well posedness of ODEs in Banach spaces under the presence of a Lipschitz condition. However, we will see that this result does not generalise to the more setting of Frechét spaces which might be surprising at first since the core of the proof – the Banach fixed point theorem – is a result for metric spaces not normed ones. Hence, we will investigate which step in the proof fails and will see that the proof breaks done solely because the Bochner inequality or triangle inequality for step functions does not hold anymore. This also shows that the integration theory due to Lebesgue and Bochner does not generalise to Frechét spaces in a straight forward way.

We begin by stating the well known local well posedness result for ODEs which we do for Banach spaces as we will see that the proof is as easy as in the Euclidean case.

1 Theorem (Picard-Lindelöf). *Let X be a Banach space and let $t_0 \in (a,b) \subseteq \mathbb{R}, x_0 \in X$ be initial values. Let further*

$$F\colon [a,b] \times \overline{B_\delta(x_0)} \to X$$

satisfy the following Lipschitz condition

$$\|F(t,x) - F(t,y)\| \leq L\,\|x-y\| \quad \text{for all } t \in [a,b], x, y \in B_\delta(x_0) \tag{1}$$

as well as

$$\|F(t,x)\| \leq K < \infty \quad \text{for all } t \in [a,b], x \in B_\delta(x_0). \tag{2}$$

Then there exists $\varepsilon > 0$ such that there is a unique solution $x\colon [t_0-\varepsilon, t_0+\varepsilon] \to X$ of the ordinary differential equation

$$\partial_t x(t) = F(t, x(t)) \quad \text{and } x(t_0) = x_0.$$

We will now discuss an example that shows that the previous result does not hold if we consider a Frechét space X instead of a Banach space.

2 Example (Non existence in Frechét spaces). Let $X = \mathcal{C}(\mathbb{R})$ be the space of continuous real functions endowed with the topology of locally uniform convergence induced by the Frechét metric

$$d(f,g) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|f-g\|_{\infty,n}}{1+\|f-g\|_{\infty,n}}, \quad \text{where } \|f\|_{\infty,n} := \sup_{x\in[-n,n]} |f(x)|\,.$$

We consider the initial value problem

$$\partial_t x(t) = x(t)^2 \quad \text{and } x(0) = \mathrm{id}_{\mathbb{R}}\,. \tag{3}$$

It is elemetary to show that the right hand side $F(t,f) = f^2$ is Lipschitz continuous with constant $L \leq 2$ and relies only on $\frac{x^2}{1+x^2} \leq 2 \cdot \frac{x}{1+x}$. Further the mapping F is bounded, since the metric is globally bounded.

We will assume that there is a local solution to the ODE (3), i.e. that there is $\varepsilon > 0$ and a curve

$$x\colon [-\varepsilon, \varepsilon] \to \mathcal{C}(\mathbb{R})$$

that satisfies (3). Since the Dirac delta distributions δ_y, i.e. the evaluations in a point $y \in \mathbb{R}$ are linear bounded functionals on $\mathcal{C}(\mathbb{R})$, they map solutions of ODEs to solutions of transformed ODEs. In our case this means that the solution x is also a pointwise solutions, i.e. that

$$\partial_t x(t)(y) = x(t)(y)^2 \quad \text{and } x(0)(y) = y.$$

Hence by the theory for real valued ODEs we have

$$x(t)(y) = \frac{y}{1 - ty}$$

given $ty < 1$ and hence for no $t \neq 0$ one obtains a function defined on whole $\mathbb{R}$.

In order to investigate which part of the proof fails in the case of Frechét spaces, we recall how the proof works in the Banach space setting.

Proof of Picard-Lindelöf. Let wlog $t_0 = 0$ and choose $\alpha \in (0, 1)$ as well as $\varepsilon := \min\left\{\frac{\alpha}{L}, \frac{\delta}{K}\right\}$. Note that by the fundamental theorem of calculus $x\colon [-\varepsilon, \varepsilon] \to X$ is a solution of the ODE if and only if it satisfies

$$x(t) = x_0 + \int_0^t F(s, x(s))\mathrm{d}s := Tx(t) \quad \text{for all } t \in [-\varepsilon, \varepsilon].$$

Hence we have transformed the ODE into a fixed point problem which we consider on

$$M = \left\{g\colon [-\varepsilon, \varepsilon] \to \overline{B_\delta(x_0)} \;\middle|\; g \text{ continuous}\right\}$$

which is a the complete metric space endowed with the topology of uniform convergence. It remains to check that T is a self mapping α-contraction.

(i) *Self mappging:* We have for all $t \in [0, \varepsilon]$ (and analogously for $t \in [-\varepsilon, 0]$)

$$\|Tx(t) - x_0\| \leq \int_0^t \|F(s, x(s))\| \,\mathrm{d}s \leq \varepsilon K \leq \delta.$$

(ii) *α-contraction:* We compute for arbitrary $t \in [0, \varepsilon]$

$$\|Tx(t) - Ty(t)\| \leq \int_0^t \|F(s, x(s)) - F(s, y(s))\| \,\mathrm{d}s \leq \varepsilon L \,\|x - y\| \leq \alpha \,\|x - y\|$$

and similarly for $t \in [-\varepsilon, 0]$.

Now the Banach Fixed point theorem yields the existence of a unique solution. □

What goes wrong in Frechét spaces

The proof of the Picard-Lindelöf theorem relies on four steps:

(i) Reformulation as a fixed point problem.

(ii) Completeness of the space of continuous functions M.

(iii) Self mapping property of T.

(iv) α-contraction property of T.

First we note that the first two steps also work in Frechét spaces.

3 Riemann integration in Frechét spaces. The whole theory of Riemann integration does not only generalise from the Euclidean case to Banach spaces but directly without big adjustments to the case of Frechét spaces. Namely for the reformulation of the ODE one needs the following properties:

(i) Riemann integrability of continuous curves.

(ii) Fundamental theorem of calculus: For a continuously differentiable curve F with $f = F'$ we have

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a).$$

On the other hand if f is a continuous curve, then

$$F(t) := \int_{t_0}^{t} f(t)\mathrm{d}t$$

is continuously differentiable with $F' = f$.

The proofs for those are either analogue to the real valued case or can easily be deduced from the real valued case by testing with bounded linear functionals.

4 Completeness of space of continuous functions. Just like in the normed case one can show that for a metric space X and a complete metric space Y the space of all bounded continuous mappings $\mathcal{C}_b(X, Y)$ is a complete metric space with the uniform metric

$$d(f, g) := \sup_{x\in X} d(f(x), g(x)).$$

A mapping into a metric space is called bounded if

$$\sup_{x_1,x_2\in X} d(f(x_1), f(x_2)) < \infty.$$

Since $\overline{B_\delta(x_0)}$ is a complete metric space this argument shows that

$$M = \left\{g : [-\varepsilon, \varepsilon] \to \overline{B_\delta(x_0)} \;\middle|\; g \text{ continuous}\right\}$$

is a complete metric space. Thus the Banach fixed point theorem would be applicable if the mapping T was a self mapping that is an α-contraction.

5 Bochner inequality. With the considerations so far, we know that one of the remaining parts, if not both have to fail. The only thing that is necessary for the proof of the self mapping as well as the contraction property is the triangle inequality for integrals

$$\left\| \int f(s)\mathrm{d}s \right\| \leq \int \|f(s)\| \,\mathrm{d}s$$

which is also called Bochner inequality. Thus this equality has to be missing in the metric case and indeed it can be seen in the proof of it where we use the scaling property that distinguishes a norm from a mapping. Indeed, for any step function $f = \sum_{i=1}^{n}(t_i - t_{i-1})x_i$ we have

$$\left\| \int f(s)\mathrm{d}s \right\| = \left\| \sum_{i=1}^{n}(t_i - t_{i-1})x_i \right\| \leq \sum_{i=1}^{n} \|(t_i - t_{i-1})x_i\| = \sum_{i=1}^{n}(t_i - t_{i-1}) \|x_i\| = \int \|f(s)\| \,\mathrm{d}s.$$

However, in the case of metric spaces we have

$$d\big((t_i - t_{i-1})x_i, 0\big) \neq (t_i - t_{i-1})d(x_i, 0)$$

and hence the proof fails for simple functions. Indeed the triangle inequality would imply

$$d(\alpha x, 0) = d\left(\int_0^\alpha x \mathrm{d}s, 0\right) \leq \int_0^\alpha d(x,0)\mathrm{d}s = \alpha d(x,0) \quad \text{for } \alpha \geq 0, x \in X.$$

However this sublinear scaling property implies for $\alpha > 0$

$$d(\alpha x, 0) \leq \alpha d(x,0) = \alpha d(\alpha^{-1}\alpha x, 0) \leq \alpha\alpha^{-1} d(\alpha x, 0) = d(\alpha x, 0)$$

and thus $x \mapsto d(x,0)$ would already be a norm.

In summary we have seen that the Bochner inequality

$$d\left(\int f(s)\mathrm{d}s, \int g(s)\mathrm{d}s\right) \leq \int d(f(s), g(s))\mathrm{d}s$$

is equivalent to the statement that $x \mapsto d(x,0)$ is a norm, i.e. that the Frechét space is indeed a Banach space.

6 Remark (Bochner integration). In a nutshell the generalisation of Lebesgue integration to Banach space valued functions $f : \Omega \to X$ with respect to a finite measure μ relies on the following steps:

(i) Consider the space of all absolutely integrable functions

$$L^1(\mu; X) = \left\{ f : \Omega \to X \;\middle|\; \int_\Omega \|f\| \,\mathrm{d}\mu < \infty \right\}$$

with the norm

$$f \mapsto \int_\Omega \|f\| \,\mathrm{d}\mu.$$

(ii) Note that the subspace of simple functions

$$\mathcal{E} = \left\{ \sum_{i=1}^n \chi_{A_i} x_i \;\middle|\; n \in \mathbb{N}, x_i \in X, A_i \subseteq \Omega \text{ measurable} \right\}.$$

(iii) Declare the integral I on the simple functions the familiar way and show the Bochner inequality for simple functions

$$\|If\| = \left\| \sum_{i=1}^n \chi_{A_i} x_i \right\| \leq \sum_{i=1}^n \chi_{A_i} \|x_i\| = \|f\|_{L^1(\mu;X)}.$$

This shows that the integral is a linear contraction from $L^1(\mu; X)$ to X and can therefore be uniquely extended.

Since we have seen that the Bochner inequality – which lies at the heart of the Bochner integration theory – is characteristic for normed spaces we note that this approach is fordoomed to fail in the case of Frechét spaces.